and maximize computational efficiency. Formulas 5) and 6) of Table I represent one-dimensional Green's functions and their Poisson summation formulas. Since these one-dimensional Green's functions do not have source singularities, they can be readily summed by direct application of the Poisson summation formula; that is, the acceleration technique defined by (1) is not required. The Poisson summation formulas for these one-dimensional Green's functions are normally only needed in certain cases involving multiple summations.

Whereas the formulas in Table I are directly applicable to one-dimensional arrays of point and line sources, they can be easily extended, by successive application, to arrays of higher dimensions involving multiple summations. This extension is a result of the property that the Fourier transform of any of these Green's functions of any one dimension can be interpreted as the Green's function at the next lower dimension. For instance, the result of applying the Poisson summation formula one time to a two-dimensional array of point sources can be interpreted as a two-dimensional Green's function. The Poisson summation formula can then be applied again to recover the final Poisson summation formula for a two-dimensional array. This procedure is demonstrated by the following example.

To obtain a summation formula for a two-dimensional array of point current sources, the Poisson summation formula is first applied to the $y$ coordinate of the three-dimensional Green's function yielding

$$\begin{aligned} f(p,q) = {} & \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \left[x^2+(y-pa)^2+(z-qb)^2\right]^{-1/2} \\ & \cdot \exp\left(-jk\left[x^2+(y-pa)^2+(z-qb)^2\right]^{1/2}\right) \\ = {} & \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \frac{1}{2\pi a} K_0\left(\left[\left(\frac{2\pi p}{a}\right)^2-k^2\right]^{1/2}\right. \\ & \left.\cdot\left[x^2+(z-qb)^2\right]^{1/2}\right) \exp\left(\frac{-j2\pi py}{a}\right). \end{aligned} \tag{2}$$

An expression equivalent to a two-dimensional Green's function can be recovered by manipulation of the above expression giving

$$\begin{aligned} = {} & \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \frac{1}{a4j} H_0^{(2)}\left(\left[k^2-\left(\frac{2\pi p}{a}\right)^2\right]^{1/2}\right. \\ & \left.\cdot\left[x^2+(z-qb)^2\right]^{1/2}\right) \exp\left(\frac{-j2\pi py}{a}\right). \end{aligned} \tag{3}$$

Applying the Poisson summation formula again, but this time to the $z$ coordinate of (3), gives the following Poisson summation formula for the Green's function $f(p,q)$:

$$\begin{aligned} = {} & \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \frac{1}{2ab}\left[\left(\frac{2\pi q}{b}\right)^2+\left(\frac{2\pi p}{a}\right)^2-k^2\right]^{-1/2} \\ & \cdot \exp\left(-|x|\left[\left(\frac{2\pi q}{b}\right)^2+\left(\frac{2\pi p}{a}\right)^2-k^2\right]^{1/2}\right) \\ & \cdot \exp\left(\frac{-j2\pi py}{a}\right)\exp\left(\frac{-j2\pi qz}{b}\right). \end{aligned} \tag{4}$$

The asymptotic form of the Green's function $g(p,q)$ and its Poisson summation formula $G(2\pi n, 2\pi q)$, required by the acceleration formula (1), are obtained from the Green's function and (4) by substituting $(x^2+c^2)$ for $x^2$.

One final comment needs to be made. At first inspection, the singularity at $n=0$ in the Poisson summation formulas of the two- and three-dimensional Green's functions of the Laplace equation, i.e., formulas 3) and 4) in Table I, seem to cause trouble. In practice, the series Green's function can always be written as the difference of two functions, both of which having the functional form of $f(n)$ in either formula 3) or 4) of Table I. With the Green's function written in this form, the $n=0$ term of the Poisson summation formula equals zero, removing the singularity and obviating the problem.

## III. Conclusion

The application of the series acceleration technique defined by (1) permits efficient computation of wide classes of problems which involve periodic sources. Many of these problems require integral transforms in the form of Poisson summation formulas which are not readily available. This paper presents a complete, convenient catalog of these Poisson summation formulas for Green's functions of the Helmholtz and Laplace equations which represent periodic sources in rectangular coordinates and homogeneous media.

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# Constant-Frequency Synthesis of Lossy Microwave Two-Ports

LODEWIJK R. G. VERSFELD

***Abstract*** **—At a fixed frequency, every linear time-invariant two-port can be described by its scattering matrix, whose elements represent eight real parameters. In this paper, it is proved that every lossy (linear, time-invariant) two-port can be canonically synthesized by eight "elementary" two-ports, which are characterized by one parameter only. Moreover, these elementary two-ports are passive and realizable in the microwave region. The synthesis is performed in the form of a cascade structure (with one "side arm" for the nonreciprocal case). Explicit formulas for the parameters of the elementary two-ports are derived.**

## I. Introduction

This paper gives a "satisfactory" synthesis of linear, time-invariant, lossy microwave two-ports at a fixed frequency. It herewith solves part of the general problem of constant-frequency synthesis of microwave networks [1]. By "satisfactory" we mean a

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TABLE I
ELEMENTARY BUILDING BLOCKS

| NAME | SYMBOL | SCATTERING MATRIX |
|---|---|---|
| ATTENUATOR | A | $S_A = \begin{bmatrix} 0 & e^{-A} \\ e^{-A} & 0 \end{bmatrix}$ with $A>0$ |
| PHASE SHIFTER | B | $S_B = \begin{bmatrix} 0 & e^{-jB} \\ e^{-jB} & 0 \end{bmatrix}$ |
| IDEAL TRANSFORMER | N | $S_N = \begin{bmatrix} r & t \\ t & -r \end{bmatrix}$ with $r^2+t^2=1$ |
| CIRCULATOR | C | $S_C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ |
| PASSIVE ONE-PORT | ρ | $\rho = \lvert\rho\rvert e^{j\varphi}$ with $\lvert\rho\rvert \leq 1$ |

synthesis using a minimum number of elementary two-ports belonging to the same class as the required two-port (in this case, the linear, time-invariant, passive class). We call a two-port elementary if it has one adjustable parameter and a simple description in terms of the scattering formalism on a power basis [2].

Here we need three types of elementary two-ports, given in Table I. They all can be considered as idealizations of realizable microwave two-ports. Furthermore, we need the three-port circulator (see Table I) for the realization of nonreciprocal two-ports.

We shall show how a lossy two-port, characterized by eight real parameters, can be synthesized with eight elementary two-ports, a, circulator, and a short circuit.

With "lossy" we mean that the two-port dissipates power for all (sinusoidal) forms of excitation. Therefore, in the passivity conditions [3], only the inequality signs appear[1]

$$\left.\begin{aligned} 1-|S_{11}|^2-|S_{21}|^2>0 \\ 1-|S_{22}|^2-|S_{12}|^2>0 \\ \left(1-|S_{11}|^2-|S_{21}|^2\right)\left(1-|S_{22}|^2-|S_{12}|^2\right)-|\bar{S}_{11}S_{12}+\bar{S}_{21}S_{22}|^2>0 \end{aligned}\right\}. \tag{1}$$

We also introduce the concept of a passive one-port, characterized by its complex reflection coefficient $\rho$ (see Table I), not for the synthesis itself, but only in order to formulate some theorems.

## II. Some Auxiliary Theorems

*Theorem 1*

Every passive reflection coefficient $\rho_0 = |\rho_0| e^{j\phi_0}$ can be realized by the circuit of Fig. 1, taking $B = -\frac{1}{2}\phi_0$ and $r = |\rho_0|$.

This can be seen by noticing that $\rho_0$ is the coefficient $S_{11}$ of the cascade of $B$ and $N$, which can be calculated easily.

*Theorem 2*

If and only if a lossless two-port is loaded on port 2 with $\bar{S}_{22}$ (i.e., the complex conjugate of its coefficient $S_{22}$), the reflection coefficient seen on port 1 equals zero.

[1]A bar above a symbol is used to denote the complex conjugate.

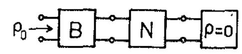

Fig. 1. Synthesis of a passive reflection coefficient.

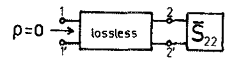

Fig. 2. Lossless two-port, loaded with $\bar{S}_{22}$.

Fig. 3. Definition of conjugate image reflection coefficients.

This can be seen by computing $\rho$ in the situation of Fig. 2 and using the properties of the $S$-matrix of a lossless two-port [4]. Likewise, loading port 1 with $\bar{S}_{11}$, yields $\rho = 0$ on port 2.

*Theorem 3*

For every lossy two-port $T$, passive reflection coefficients $\rho_1$ and $\rho_2$ can be found such that the situations of Fig. 3 hold.

This is the concept of conjugate image impedances [5]–[7] translated into the scattering formalism. The straightforward calculation of $\rho_1$ and $\rho_2$ yields

$$\rho_1 = \frac{1}{2E_1}\left(D_1 \pm \sqrt{D_1^2 - 4|E_1|^2}\right) \tag{2a}$$

$$\rho_2 = \frac{1}{2E_2}\left(D_2 \pm \sqrt{D_2^2 - 4|E_2|^2}\right) \tag{2b}$$

where (with $\det S = S_{11}S_{22} - S_{12}S_{21}$)

$$D_1 = 1 + |S_{11}|^2 - |S_{22}|^2 - |\det S|^2$$
$$E_1 = S_{11} - \overline{S_{22}} \det S$$
$$D_2 = 1 + |S_{22}|^2 - |S_{11}|^2 - |\det S|^2$$
$$E_2 = S_{22} - \overline{S_{11}} \det S.$$

These formulas can be obtained also [8], [9] by changing the normalization impedances of a given $S$-matrix such that in the new $S$-matrix $S_{11} = S_{22} = 0$ holds, i.e., both ports are simultaneously conjugate matched. It can be proved by rather complicated algebraic manipulations that (2a) and (2b) have the following properties:

a) $D_1^2 - 4|E_1|^2 = D_2^2 - 4|E_2|^2$.
b) The modulus of the product of the two solutions for $\rho_1$ equals unity. The same holds for $\rho_2$.
c) $D_1^2 - 4|E_1|^2$ is positive for lossy two-ports.

Properties b) and c) together imply the existence of one passive (and one active) solution for $\rho_1$. Because of property a), the same holds for $\rho_2$. Moreover, it can be proved that $D_1$ and $D_2$ are positive for lossy two-ports. This implies that in (2a) and (2b), the passive solutions are obtained by taking the minus signs.

*Theorem 4*

In the cascade of Fig. 4, with $T$ an arbitrary lossy two-port, the two-ports $N_1, B_1, B_2, N_2$ can be chosen such that for the resulting two-port (i.e., between ports 1 and 4) $S_{11} = S_{22} = 0$.

*Proof:* Let $\rho_1$ and $\rho_2$ denote the reflection coefficients belonging to $T$, as meant in Theorem 3.

According to Theorem 1, $B_2$ and $N_2$ can be chosen such that loading port 4 with $\rho = 0$, the reflection coefficient on port 3,

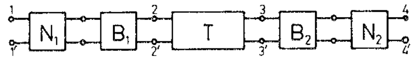


Fig. 4. Transformation of a two-port into one having zero input reflection coefficients.

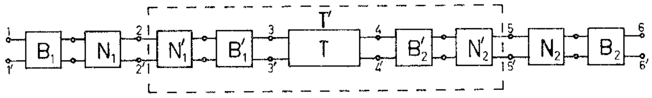


Fig. 5. Transformation of a two-port into an equivalent one.

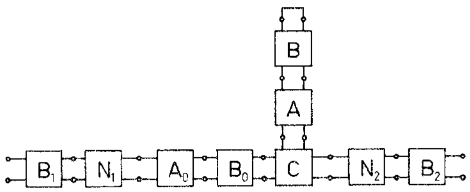


Fig. 6. Synthesis of a lossy two-port using eight elementary two-ports.

seen to the right, equals $\rho_2$. Likewise, $B_1$ and $N_1$ are taken such that loading port 1 with $\rho = 0$ yields $\rho = \rho_1$ on port 2 (seen to the left).

Now, loading port 4 with $\rho = 0$ (and looking to the right all the time), we see on port 3: $\rho = \rho_2$; on port 2: $\rho = \overline{\rho_1}$ (Theorem 3!); on port 1: $\rho = 0$ (Theorem 2!). On the other hand, we see on port 1, by definition, the reflection coefficient $S_{11}$ of the resulting two-port. So, $S_{11} = 0$. Using the same reasoning from left to right yields the result $S_{22} = 0$.

## III. The Synthesis of Lossy Two-Ports

In the cascade connection of Fig. 5, $T$ is the two-port to be synthesized. Furthermore, let the following relations exist:

$$B_i = -B_i' \quad r_i = -r_i', \qquad i = 1,2. \tag{3}$$

$B_i, B_i', r_i, r_i'$ in this expression are the scattering parameters (according to Table I) of two-ports $B_i, B_i', N_i, N_i'$, respectively.

From (3), it is clear that there is a through-connection between ports 1 and 3 as well as between ports 4 and 6. Therefore, the resulting two-port is identical to two-port $T$. The synthesis will relate to this resulting two-port. Using Theorem 4, we choose $N_1', B_1', B_2', N_2'$ such that $S_{11}' = S_{22}' = 0$ for two-port $T'$ in Fig. 5. So

$$S_{T'} = \begin{bmatrix} 0 & S_{12}' \\ S_{21}' & 0 \end{bmatrix}.$$

This two-port $T'$ can be realized by a cascade of an attenuator, a phaseshifter, and a circulator, the latter being loaded on its third port with the short-circuited cascade of an attenuator and a phase shifter. Applying this to the cascade of Fig. 5 yields the final result given in Fig. 6.

Here we see how a lossy two-port, given by eight real parameters, is synthesized with a circuit of eight elementary two-ports, having together eight real parameters as well.

The last step is to express the latter parameters into the former ones. Combining (3) with Theorems 1 and 4 yields

$$\left.\begin{aligned} B_1 &= \tfrac{1}{2} \arg \rho_1 \\ r_1 &= |\rho_1| \\ B_2 &= \tfrac{1}{2} \arg \rho_2 \\ r_2 &= -|\rho_2| \end{aligned}\right\} \tag{4}$$

$\rho_1$ and $\rho_2$ being given by (2a) and (2b), respectively.

Referring to Fig. 5, matrix $S_{T'}$ can be calculated from the cascade of $N_1', B_1', T, B_2', N_2'$. In addition, using (3) and (4), we find

$$S_{12}' = S_{12} \frac{e^{j\frac{1}{2}(\arg \rho_1 + \arg \rho_2)}}{1 - \rho_1 S_{11}} \sqrt{\frac{1 - |\rho_1|^2}{1 - |\rho_2|^2}}. \tag{5}$$

Since $S_{12}/S_{21}$ does not change if reciprocal two-ports are put in cascade with a given two-port, we get

$$S_{21}' = \frac{S_{21}}{S_{12}} S_{12}'. \tag{6}$$

The parameters of the remaining two-ports of Fig. 6 can be expressed easily in $S_{12}'$ and $S_{21}'$ as follows.

a) If $|S_{21}'| \leqslant |S_{12}'|$:

$$\left.\begin{aligned} e^{-A_0} &= |S_{12}'| & B_0 &= -\arg S_{12}' \\ e^{-A} &= \sqrt{\left|\frac{S_{21}'}{S_{12}'}\right|}\ (\leqslant 1) & B &= -\tfrac{1}{2}\left(\pi + \arg S_{21}' - \arg S_{12}'\right) \end{aligned}\right\} \tag{7}$$

the circulator has to be used with port 1 on the left-hand side and port 2 on the right-hand side (see Table I).

b) If $|S_{21}'| \geqslant |S_{12}'|$:
Formulas (7) with $S_{12}'$ and $S_{21}'$ interchanged; the circulator now has to be used with ports 1 and 2 interchanged.

Notice that in the reciprocal case, due to the values $e^{-A} = 1$ and $B = -\frac{1}{2}\pi$, the circulator turns into a through-connection.

## IV. Conclusion

At a fixed frequency, every linear, time-invariant lossy two-port can be canonically synthesized by the structure of Fig. 6.

Whether or not this realization is unique is an open question. Other possible structures are

- the "cascade with side arm," not having the same number and order of attenuators, phase shifters, and ideal transformers as the structure of Fig. 6,
- combinations of cascade, series, and parallel structures.

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